

# First-person visualizations of the special and general theory of relativity

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August 29, 2007

Accepted for publication in *The European Journal of Physics*

## **Abstract**

Visualizations that adopt a first-person point of view allow observation and, in the case of interactive simulations, experimentation with relativistic scenes. This paper gives examples of three types of first-person visualizations: watching objects that move at nearly the speed of light, being a high-speed observer looking at a static environment and having a look-around near a compact object. I illustrate and explain the main aspects of the visual observations, outline their use in teaching relativity and report on teaching experiences. For teaching purposes, our visualization work is available on the web site [www.spacetime.travel.org](http://www.spacetime.travel.org) and its German counterpart [www.tempolimit-lichtgeschwindigkeit.de](http://www.tempolimit-lichtgeschwindigkeit.de). This paper assumes some basic knowledge about relativity on the part of the reader. It addresses instructors of physics at the undergraduate and advanced secondary school level as well as their students.

(Some figures in this article are in colour only in the electronic version)

## **1 Introduction**

The theory of relativity started out with the reputation of being a particularly difficult and abstract theory. Anecdotes about this topic are widely known such as Einstein's question 'Why is it that nobody understands me, yet everybody likes me?'<sup>1</sup> or the story about Sir Arthur Eddington who, when somebody called him one of the three men in the world who really understood the theory of relativity, replied that he did not know who might be the third.

Today, relativity theory is widely taught at universities and increasingly finds its way into secondary school curricula. Roman Sexl analysed this transition from a seemingly incomprehensible theory to a widely taught one (Sexl 1980) and explained it by increased faith in the theory and better verbal explanations of the relevant effects. His conclusion in 1980 was that for further progress in teaching relativity, suitable audiovisual media should be developed.

In this contribution I will focus on one type of visual teaching aides, namely visualizations that adopt a first-person point of view. Many difficulties in understanding the theory of relativity come from the fact that relativistic effects are not part of everyday experience and may even seem to contradict this experience. Also, direct observation by way of demonstration experiments generally is not possible. First-person visualizations are meant to fill this gap. Simulated movies allow 'observations' and interactive computer simulations provide virtual experience of relativistic scenes.

The results of these simulations are often perplexing, even for experts in the field. This is well illustrated by early textbook illustrations of the visual appearance of objects moving at nearly the speed

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<sup>1</sup>Einstein, 1944.

of light: they were wrong (Gamow 1940 (corrected in Gamow 1961), Fuchs 1965). Only from 1959 onwards, a series of investigations clarified what such an object would really look like (Penrose 1959, Terrell 1959; an earlier study by Lampa (1924) remained largely unnoticed). The detailed explanation of the phenomena by means of sketches and animations is therefore essential for teaching purposes, and we put equal emphasis on computing the simulations and on providing explanatory material.

In the following, I will illustrate the main aspects of these first-person visualizations with several examples, outline their use in teaching relativity and report on teaching experiences in university and secondary school.

## 2 Visualizations of the special theory of relativity

Special relativistic effects are minute in everyday life, but become dramatic when velocities close to the speed of light are involved. The visualizations therefore explore high-speed motion. This is done from two different points of view: we either watch high-speed objects moving by or we move ourselves at nearly the speed of light through a static scene.

By the principle of relativity, these two points of view are perfectly equivalent, i.e. given the same relative velocity, the resulting movie is the same. The explanation, however, is quite different in the two different reference frames, as outlined below: the flight times of light signals explain the visual effects for high-speed objects, while high-speed motion of the observer provides a striking illustration of the aberration of light.

There are three different approaches to the choice of length and time scales as follows. Either the objects have dimensions of the order of light seconds like stars or large planets, then the virtual fly by takes some seconds and can comfortably be recorded by a camera. Or the scene is taken from everyday life, e.g. houses or trains, in which case the fly by time would be extremely short. We can then either invoke an improbably fast high-speed camera (in line with the equally improbable high-speed motion) and play the film in extremely slow motion or, following Gamow (1940), we can simulate a virtual world with a suitably reduced speed of light of the order of everyday velocities. In our examples we freely use either stellar or terrestrial objects, as needed to make the respective simulation both appealing and instructive.

### 2.1 Moving object

The example shown in figure 1 exhibits the main particularities of objects watched in high-speed motion. A number of dice is set up in a row (fig. 1(a), at rest). In the simulation, these dice move in a single file at 95% of the speed of light, the face with the '3' being in front and the '4' in the rear. According to special relativity, the whole arrangement suffers length contraction by the factor  $f = \sqrt{1 - v^2/c^2}$ ,  $v$  being the velocity of the dice and  $c$  being the speed of light; in this example  $f = 0.3$ . One might therefore expect to *see* the dice shortened (fig. 1(b)), and the early, incorrect textbook illustrations were in fact pictures of this kind. The correctly computed looks of the moving dice are shown in figure 1(c). The moving dice appear primarily rotated. Also, the vertical edges appear slightly bent, most notably in the dice that is closest to the observer.

In this computation, the finite velocity of the light coming from the dice has been taken into account. As illustrated in figure 2, when a point has coordinates  $(x, y, z)$  at the time of observation, then the light observed at that instant, having spent a certain time to reach the observer, must have been emitted earlier at the previous position  $(x_p, y, z)$  (where the  $x$ -axis here is in the direction of motion). A straightforward calculation shows that the apparent position is given by

$$x_p = \gamma^2 x - \beta \gamma \sqrt{\gamma^2 x^2 + y^2 + z^2} \quad (1)$$

with  $\beta = v/c$ ,  $v$  being the velocity of the object and  $c$  being the speed of light,  $\gamma = 1/\sqrt{1 - v^2/c^2}$ .

The time of flight affects not only the apparent position, but also the apparent shape of an object. Different points on the surface of the object are at different distances from the camera. Hence, photons reaching the observer simultaneously have not been emitted simultaneously by all points of the object.

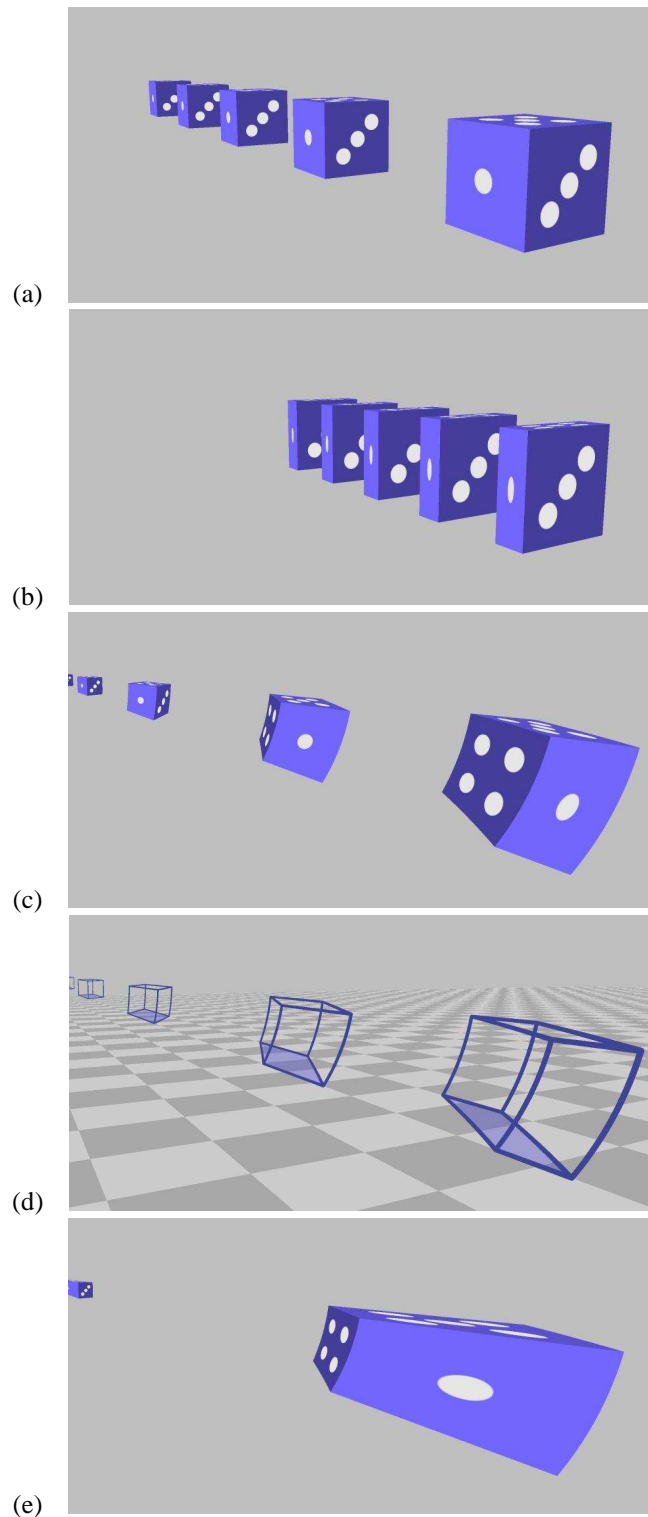


Figure 1. A row of dice ((a), at rest) moves in a single file at 95% of the speed of light (motion from left to right). The moving dice are length contracted, so that one might (wrongly) expect them to look as shown in (b). In the visual observation, the dice appear rotated (c). However, when some perception in depth is provided ((d), skeleton cubes sliding over a plane with their 'footprints' marked) the dice are seen as sheared rather than rotated. (e) The predicted 'classical' appearance of the dice (no length contraction) is shown. (All images with a  $54.4^\circ$  horizontal opening angle.) A short movie of (c) of this figure is attached to the online version of the article (MPEG 1, 3 MB).

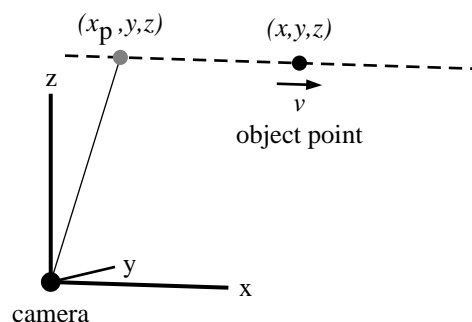


Figure 2. A moving point source has coordinates  $(x, y, z)$  at the instant when the recording camera is released. The camera records light that has been emitted at an earlier position  $(x_p, y, z)$  (motion is in  $x$ -direction).

The set of all the emission points  $(x_p, y, z)$  is therefore a distorted image of the object surface and what we see is the projection of this distorted ‘phantom object’.

Terrell (1959) has shown that for objects subtending a small solid angle, the projected image agrees with the projection of the suitably rotated object at rest. This is often phrased as ‘a fast moving object appears rotated’. However, this statement has been a point of some debate, since the phantom object itself is sheared rather than rotated and opinions differed about whether or not a snapshot would convey the impression of shear (Mathews and Lakshmanan 1972, Sheldon 1988, Terrell 1989).

We find that in the visualization, either impression can be evoked. When the cubes are isolated objects as in figure 1(c), they clearly appear rotated. We can, however, make the shear visible if we change the scene to provide a better perception in depth. In figure 1(d) the cubes slide over a plane and, in addition, are hollow with the sides cut away so that only the edges remain and the ‘footprints’ on the plane are visible. This image gives a visual impression of the sheared phantom cubes. While creating these simulations, it became apparent that it is not easy to make the cubes look sheared. Much guidance to the eye must be provided to dispel the prevailing impression of rotation.

In a sense, the time-of-flight effects are classical effects that could have been considered before the formulation of special relativity. A comparison with such a ‘classical’ picture (figure 1(e)) highlights the relativistic aspects: the time-of-flight effects make the cubes appear elongated (when approaching). Remarkably, the correct picture, including the Lorentz contraction, is less distorted than its classical counterpart. So, in a way special relativity makes fast looking objects look less strange than the classical theory would.

A particularly surprising feature of these simulations is the visibility of the rear side of the cubes (the ‘4’). At the first sight it seems to be impossible for photons from the rear to reach the observer, since the cube is in between and should block the way. As is illustrated in figure 3, the photons escape because the cube moves out of the way fast enough. The velocity vector of a photon directed towards the observer can be split up into a horizontal component (towards the cube) and a vertical component. Each component by itself is smaller than the speed of light. The horizontal component can, therefore, be surpassed by the cube velocity. In this case, the cube outruns the photon and the distance between the photon and the cube increases while the cube moves to the right and the photon towards the observer; the photon escapes.

Visualizations of this kind can serve different purposes in teaching. They explore consequences of the finite speed of light that are fascinating to most students. In the context of relativity they clearly show up the difference between visual observations and measurements and thus help to point out the need for careful definitions of measurement procedures – these are basic to the understanding of relativity but the need is not easily appreciated by many students. The topic also lends itself to student projects: simulations of simple objects, e.g. wire-frame models of cubes, lattices or spheres, can be created with basic programming skills. Shirer and Bartel (1967) report on a freshman project of this kind. Another project that I worked on with advanced secondary school students is the construction of a set of real wire-frame models: of an object, its length-contracted counterpart and phantom objects for visual observations from

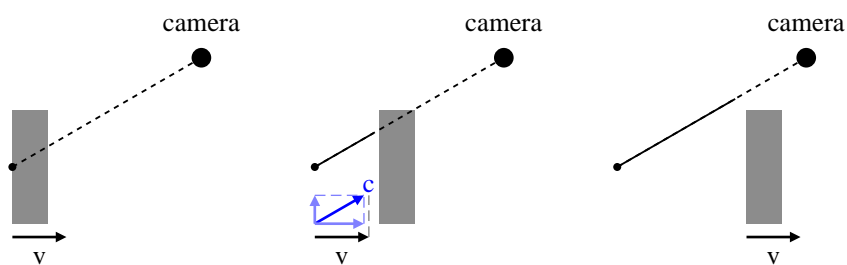


Figure 3. A photon emitted from the back side of the moving cube may escape in a forward direction, provided that the velocity component towards the cube is less than the cube velocity.

different directions. Looking at the phantom object model from the right distance then produces the visual impression of the object in high-speed motion. The topic is also relevant in astronomy. Motion at nearly the speed of light is observed in some cases, most notably quasars and gamma burst sources. Some of the visual effects that appear in the simulations must be taken into account in order to interpret the observations (Kraus 2005a).

## 2.2 Moving observer

The visual perception of a moving observer is best demonstrated in comparison with an observer at rest *at the same place*. Figure 4 shows two snapshots taken from a simulation using as setting a detailed three-dimensional model of the old city centre of Tübingen<sup>2</sup>. In the simulation, the speed of light in ‘virtual Tübingen’ is reduced so that we experience relativistic effects while riding a bike through the city (Borchers 2005, Kraus and Borchers 2005).

Figure 4(a) is a snapshot taken while standing in the Kornhausstraße and looking down the street towards the Hirschgasse. Note the bakery on the left hand side and the ice cream parlour at the end of the road.

The second snapshot, figure 4(b), is taken while driving down the road at 95% of the speed of light. The camera is released at the instant when we pass the spot where the first snapshot was taken. We now see the bakery far in front of us, the ice cream parlour at the end of the road is hardly visible and close to us the edges of the houses appear bent. Next to the bakery, two more houses appear in the picture which are actually standing to our side and behind us.

Aberration is the reason for the differences between the two snapshots. A light ray that reaches a stationary observer at an angle  $\theta$  to the direction of motion (where  $\theta = 0$  is a ray arriving head on) is perceived by the moving observer at an angle  $\theta'$  with

$$\cos \theta' = \frac{\cos \theta + \beta}{1 + \beta \cos \theta}, \quad (2)$$

where  $\beta = v/c$ ,  $v$  being the velocity of the observer and  $c$  being the speed of light.

Figure 5 illustrates this effect. A number of light rays is shown that reach the stationary observer evenly from all sides. The directions that the moving observer attributes to these same light rays are shifted towards the front. She therefore perceives objects in front of her within a smaller solid angle which makes them appear to be farther away. Also, light rays which reach the stationary observer obliquely from behind may to the moving observer come obliquely from her front; for this reason, a high-speed camera can image objects located behind its back.

One may ask how precisely light that is coming obliquely from behind is supposed to enter the moving camera that is looking forward all the time. The answer is illustrated in figure 6 with a pinhole camera: a photon approaching obliquely from behind moves in front of the camera, is caught by the pinhole and is then swept up by the image plane.

<sup>2</sup>We thank Professor H Bülthoff for his kind permission to use this model that has been constructed at the Max Planck Institute for Biological Cybernetics, Tübingen.

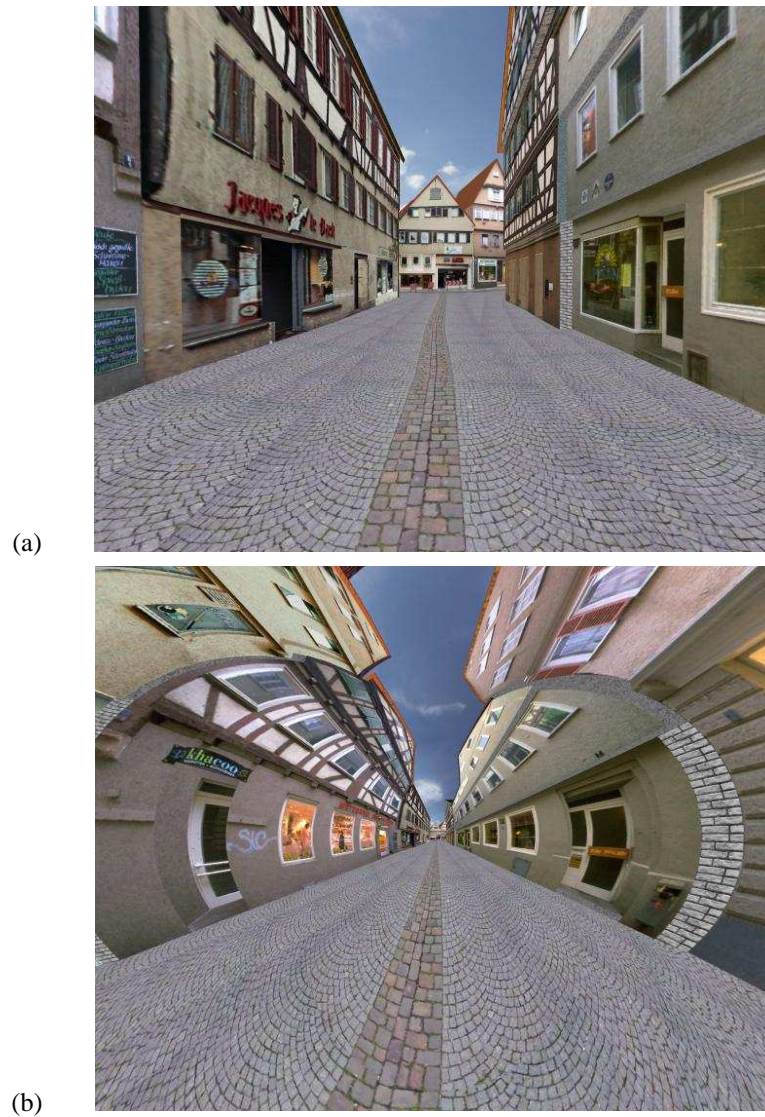


Figure 4. Looking down the Kornhausstraße towards the Hirschgasse in Tübingen. (a) Snapshot while at rest. (b) Snapshot while moving down the road at 95% of the speed of light, looking forward. Both snapshots are taken *at the same place*. The opening angle of the camera is  $90^\circ \times 112.5^\circ$ , i.e. these are wide-angle images. In order to obtain the visual impression that one would have with the naked eye, one would have to scale up the picture to DIN A1 format and then look at it from a distance of 30 cm. Computer simulation by Marc Borchers. For two movies of this simulation, see the online version of Kraus and Borchers (2005).

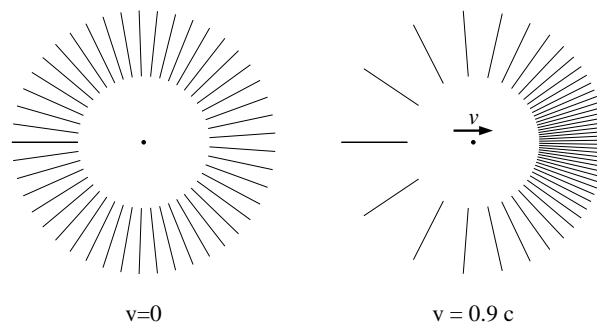


Figure 5. Aberration. Left: a number of light rays reaching a stationary observer evenly from all sides. Right: a moving observer at the same place attributes different direction to the same light rays.

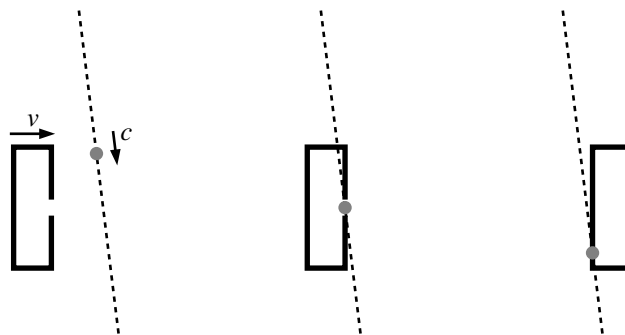


Figure 6. A moving camera can image objects located behind its back. The pinhole camera moves from left to right at 95% of the speed of light. A photon approaching from above and behind the camera is caught by the pinhole and then swept up by the image plane.

Apart from snapshots and movies, the Tübingen simulation also allows interactive manipulation. The user determines speed and direction, while the apparent view of the city is displayed in real time. The interactive control can be exercised via a joystick. There is also an ‘exhibition version’ simulating a bike ride, where the user sits on an exercise bike. Turns of the handle bar and pedalling determine direction and speed respectively. The apparent view of the street scene is projected onto a large screen in front of the bike.

There are features of the simulation that are best experienced in an interactive form, for example the effect of acceleration. Whenever one accelerates, the houses in front seem to move into the distance; this is reversed upon slowing down. The reason is that with increasing velocity, the aberration becomes stronger and objects seen in front take up a smaller solid angle. Because of looking smaller, they seem to be farther away. The use of a similar interactive simulation in an undergraduate relativity class has been described by Savage et al. (2007).

The visualizations of high-speed motion can be used to illustrate the principle of relativity. A snapshot taken at high relative velocity between the object and the observer may be computed in one of two ways: either the object is considered to be moving and its appearance is the projection of the respective phantom object or the observer is considered to be moving and his visual perception is determined by applying the aberration formula. Both methods give the same result as one can deduce from equations (1) and (2). It is instructive to compare this to the non-relativistic case. Figure 7 shows again a row of dice ((a), at rest), here seen with the line of sight perpendicular to the direction of motion. In the correctly computed image, the dice appear rotated (figure 7(b)). These may either be moving dice watched by a stationary observer or vice versa. The ‘classical’ moving dice (figure 7(c)), computed without the Lorentz contraction, appear both rotated and elongated. In contrast, the ‘classical’ moving observer whose snapshot is computed using the non-relativistic aberration formula sees the dice rotated

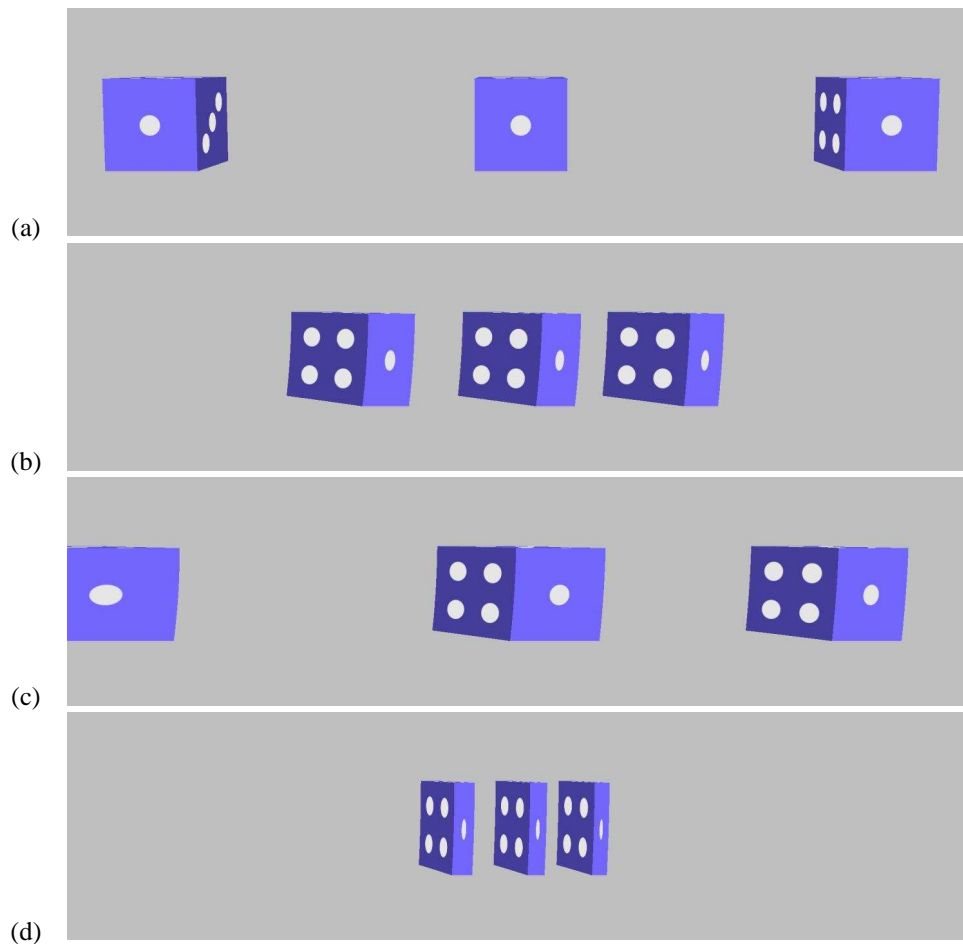


Figure 7. Three dice ((a), at rest) are watched at a relative velocity of 90% of the speed of light. The direction of motion is perpendicular to the line of sight. (b) The dice moving from left to right or the observer from right to left, the visual appearance is the same. (c) ‘Classical’ computation of the dice moving from left to right (no length contraction), (d) “Classical” computation for an observer moving from right to left (non-relativistic aberration formula). (All images with a  $54.4^\circ$  horizontal opening angle.)

and contracted (figure 7(d))!

The simulation of a high-speed observer’s perception is a programming project that is not too difficult for a simple scene consisting e.g. of geometric objects (for a sample programme see Kraus and Zahn (2003)).

These visualizations may also be of interest in teaching astronomy. Aberration is an important phenomenon in astronomy, and the visual observations of a fast moving observer provide impressive illustrations.

### 3 Visualizations of the general theory of relativity

The general theory of relativity is a theory of gravitation. First-person visualizations in this context may explore visual observations in high-gravity surroundings. Typical examples are observations near neutron stars or black holes as in the early work by Luminet (1979) and Nemiroff (1993). In the vicinity of compact objects, gravitational light deflection is strong and generates surprising visual effects.

To illustrate this, figure 8 shows some snapshots from a virtual journey towards a black hole (Kraus



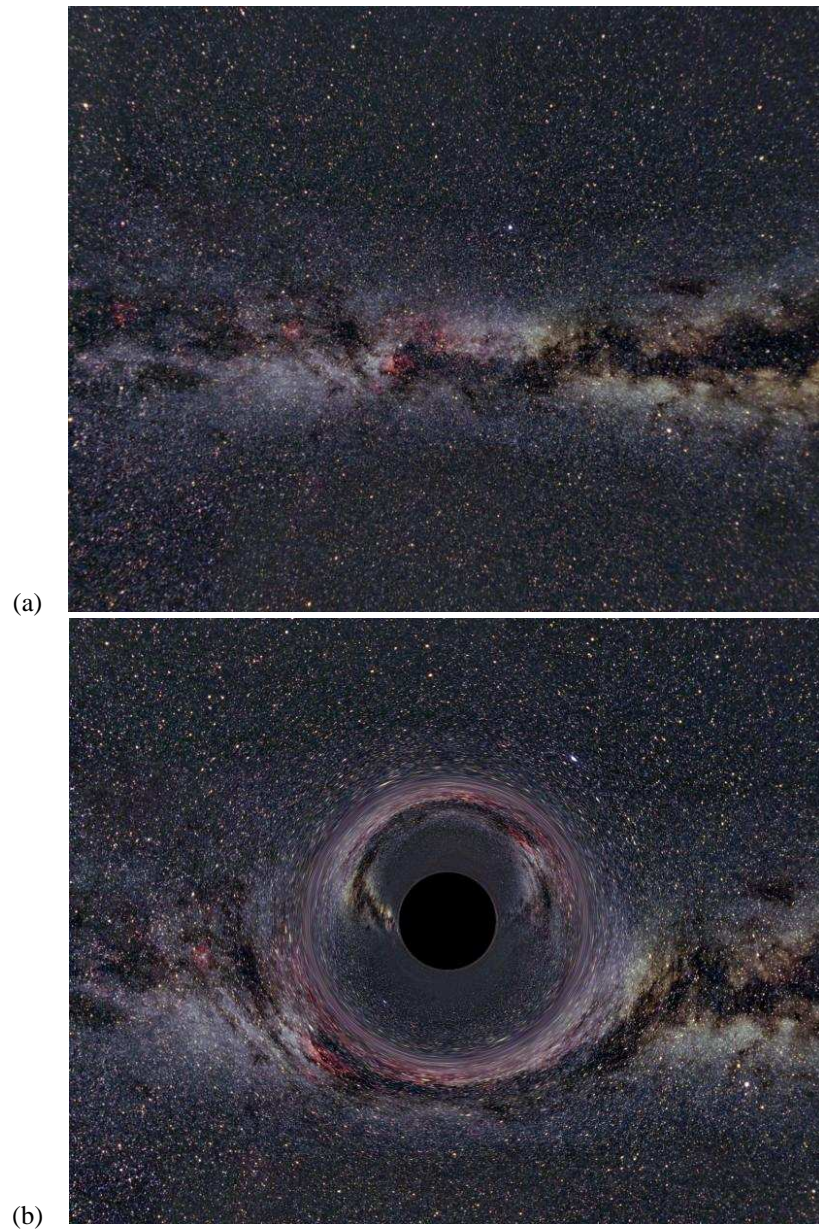


Figure 8. A black hole of ten solar masses in front of the Milky Way (a), seen from a distance of 600 km (b), 45 km (c) and 13 km (d) above the event horizon. In (b) we look towards the black hole, in (c) to the side, in (d) away from it. (All images with a  $90^\circ$  horizontal opening angle.)

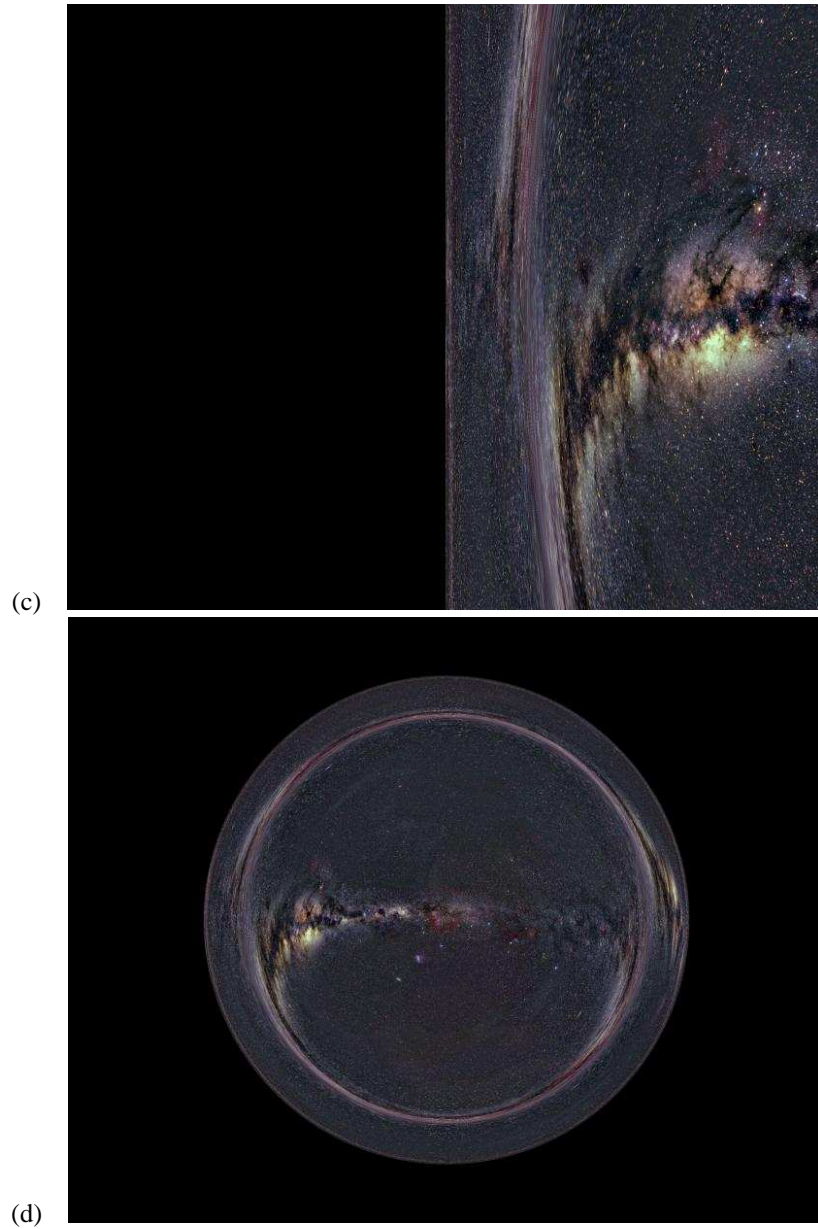


Figure 8 (Continued.)

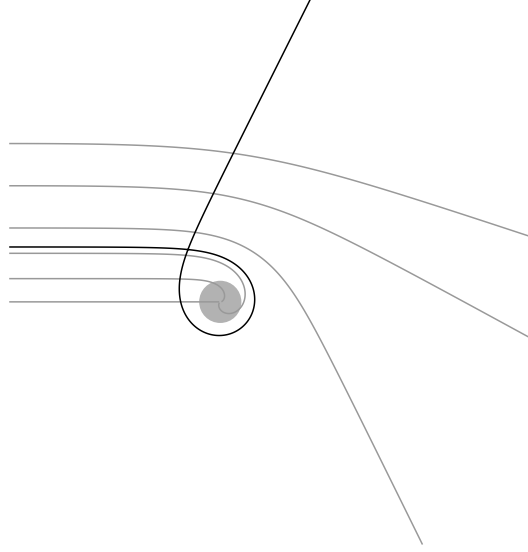


Figure 9. Photon orbits near a black hole. The region  $r < r_s$  ( $r_s$  the Schwarzschild radius) is shaded in grey. Close to the photon radius  $1.5 r_s$ , the deflection may become very large.

2005b). The numbers given apply to a black hole of ten solar masses (Schwarzschild radius  $r_s = 30$  km).

In the visualization, we place the black hole in front of the Milky Way<sup>3</sup> (figure 8(a)). Each snapshot is taken while the observer keeps his distance to the black hole constant, i.e. while he is hovering near the black hole. From a distance of 600 km (radial coordinate  $r = 20 r_s$ ), the Milky Way in the background appears distorted with arc-like structures and double images (figure 8(b)) close to the central black disc from which no light is received. Farther in, the black disc gets larger and larger until it completely fills the front view. The camera is then turned to the side. 45 km above the event horizon ( $r = 1.5 r_s$ ) it records the image of figure 8(c), in which the black disc covers exactly half the sky. Moving further in, the black disc grows still larger and finally fills the side view as well. The camera is then turned to the back and at 13 km above the horizon ( $r = 1.05 r_s$ ) records figure 8(d): the observer seems to be engulfed by the black hole, the night sky is reduced to a small round patch, diminishing ever more as the observer approaches the horizon of the black hole.

In these simulations, the photon paths  $x^\mu(\lambda)$  from the background image to the observer are computed as solutions of the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \sum_{\nu, \kappa=0}^3 \Gamma_{\nu\kappa}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\kappa}{d\lambda} = 0 \quad \text{subject to} \quad \sum_{\mu, \nu=0}^3 g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \quad (3)$$

with the spacetime metric  $g_{\mu\nu}$  and the Christoffel symbols  $\Gamma_{\nu\kappa}^\mu = 0.5 \sum_{\sigma=0}^3 g^{\mu\sigma} (\partial g_{\nu\sigma} / \partial x^\kappa + \partial g_{\kappa\sigma} / \partial x^\nu - \partial g_{\nu\kappa} / \partial x^\sigma)$ . This is a set of four coupled ordinary differential equations which is solved numerically.

Close to a black hole, gravitational light deflection is a dramatic effect as is illustrated in figure 9; near the photon radius at radial coordinate  $r = 1.5 r_s$ , photons are so strongly deflected that they may circle the black hole, even several times, before either escaping or crossing the event horizon. The photon radius itself is an unstable circular photon orbit that forms the borderline between photon orbits that extend to infinity and those that end up in the black hole.

This explains the peculiar observation that the black hole appears to cover half the sky (figure 8(c)) when the observer is still some way outside the horizon. The image is a snapshot taken when the observer is located on the photon radius. In this position, when looking exactly to the side ( $90^\circ$  to the radial direction) the observer receives photons moving on the circular orbit, i.e. coming from his own back of the head. Only photons that arrive from outside of the circular orbit come from the exterior of

<sup>3</sup>This background image is the All-Sky Milky Way Panorama by Axel Mellinger (Mellinger 2000, 2005).

the black hole. The exterior region, therefore, covers exactly half the observer's sky and the black disc the other half.

For teaching purposes, an interactive black hole simulation offers the possibility of experimenting with light deflection (Zahn and Kraus 2006). In this simulation, a black hole is placed in between the observer and a background image, with a large distance to both, and the user can move the black hole across the image interactively. This may be used to demonstrate several astrophysically important effects: changes in the apparent position (as observed in stars near the sun during a solar eclipse, one of the classical tests of general relativity), arc-like deformations (as observed in gravitational lensing of galaxies) and the formation of an Einstein ring. The simulation also clearly shows the large increase in brightness when an Einstein ring is formed, an effect which is exploited to search for dark matter in the Milky Way in the shape of compact halo objects (Paczynski 1996).

In courses on general relativity, photon orbits in a Schwarzschild spacetime are a standard topic and visualizations such as those described above may be used as complements to the usual analytical treatment.

In astronomy, gravitational light deflection has increasingly become an important topic. While at the beginning of the twentieth century it seemed to be an exotic phenomenon, observed in order to test a new theory of gravitation, it is now a tool used routinely in the study of cosmic objects ranging from extrasolar planets to the structure of galaxies. Visualizations can illustrate many of these astronomical applications (Kraus 2006).

All the visualizations shown in this contribution focus on geometric effects. In order to make the geometry clearly visible, Doppler and searchlight effects have not been taken into account. However, these effects are dramatic in both high-speed motion and strong gravity environments (Kraus 2005a). In order to include them, the spectrum of the objects must be known, not only in the visible range but also beyond. To compute the observed colour from the suitably transformed spectra is an unusual and interesting application of colorimetry (Kraus 2000).

## 4 Teaching use

Visualizations of relativistic effects are a complement to the standard approaches to the special and general theory of relativity. The pictures and movies serve a number of purposes: as an introduction, they are sure to arouse interest. We have found that objects and scenes from everyday life work best in this respect. Secondly, by showing extreme situations, the images illustrate effects in a dramatic and highly visible way, aiding long-term memory. Thirdly, and maybe most importantly, they act as a counterbalance to wrong or mistaken pictures and concepts. Examples are widely used pictures of black holes (Berry 2001, 2003) or common misinterpretations of terms like 'observing' which many students readily equate with 'observing visually'. Images that are both impressive and correct are the best means to replace misconceptions of this kind with more correct notions.

In undergraduate education, we consider these visualizations to be especially useful when teaching relativistic physics to students who will continue as physics teachers, because our experience shows that this material can very profitably be utilized in secondary schools. The travelling exhibition 'Einsteinmobil' (see <http://www.einsteinmobil.de> for details) has been bringing interactive simulations and movies into secondary schools since the beginning of 2006. The exhibits stay for a typical duration of 2 weeks and are used at the discretion of the school. To evaluate the project, teachers of the participating schools are asked to complete questionnaires. A general finding is that most or all students show much interest, independent of age, of gender and of their general interest in physics.

The visualizations of our group are being used in a wide range of teaching situations from undergraduate university classes and physics teachers seminars to secondary school projects, planetarium shows and museum exhibitions. Vollmer (2006) even describes a project in primary school.

In order to make the visualizations available to both instructors and students, we maintain the web site <http://www.spacetime.travel.org> (with its German counterpart <http://www.tempolimit-lichtgeschwindigkeit.de>) on which we present a collection of multimedia contributions, including many short simulation movies. The contributions range from entertaining introductions (e.g. relativistic soccer) to more theoretical papers on an advanced undergraduate level. For use

by instructors, the simulation movies are provided in a quality suitable for projection in class. This site will be updated with future visualization projects.

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